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Growth in the Robinson–Solow–Srinivasan model: Undiscounted optimal policy with a strictly concave welfare function

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Abstract

In a special case of a model due to Robinson, Solow and Srinivasan, we characterize the optimal policy function (OPF) for undiscounted optimal growth with a strictly concave felicity function. This characterization is based on an equivalence of optimal and minimum value-loss programs that allows an extension of the principal results of dynamic programming. We establish monotonicity properties of the OPF, and obtain sharper characterizations when restrictions on the marginal rate of transformation are supplemented by sufficient conditions on the "degree of concavity" of the felicity function. We show that important similarities and intriguing differences emerge between the linear and strictly concave cases as the marginal rate of transformation moves through its range of possible values.

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1. Introduction

In this paper, we re-examine a model of development planning originally formulated by Robinson (1960); Solow (1962); Srinivasan (1962) (henceforth, the RSS model). The model was used notably by Stiglitz (1968) to provide some policy prescriptions about the optimal choice of techniques for economic development. Robinson (1969) commented on Stiglitz' solution by criticizing his assumption of a fixed positive discount rate, continuous time and the linearity assumption in the specification of the planner's felicity function.

In recent work (see Khan and Mitra, 2005), keeping in mind the Robinson criticisms, we reformulate the problem as a particular instance of the general theory of optimal intertemporal allocation in the undiscounted case, as presented by Gale (1967); McKenzie (1968); Brock (1970). Unlike Stiglitz, we treat the original Ramsey problem (by directly dealing with the undiscounted case, using a version of the overtaking criterion to formulate the appropriate notion of optimality), and we use a discrete-time formulation. It turns out that the policies proposed by Stiglitz are optimal for some choice of parameter values, but not so for others. We have also shown (with an example) that the adoption of a

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strictly concave felicity function can make the optimal choice of technique in the short-run quite different from that in the long-run. Our paper showed that Robinson's criticisms were entirely justified and that the general theory of optimal intertemporal allocation can be used to improve considerably our understanding of these models of development planning.

In the course of this research, and somewhat to our surprise, we realized that the RSS model in turn had much to contribute to the understanding of the general theory of intertemporal allocation. In fact, a stripped-down two-sector version of the RSS model, in which the optimal choice of technique was not an issue any more (since there is only one type of machine), provided considerable insight about the phenomenon of non-uniqueness of optimal programs, the connection between optimality and value-loss minimization, and the relationship between the value-loss method and dynamic programming in the undiscounted case (see Khan and Mitra, in press-a,b).

The current paper can be viewed as a continuation of the two strands of our research discussed above. In keeping with the second strand of research, we examine a two-sector RSS model, but in contrast to Khan and Mitra (in press-a,b), we treat explicitly the case of a strictly concave felicity function. We hope, thereby, to accomplish two things. First, this would be a natural robustness test on the results obtained by us in the case of a linear utility function. Second, we hope that the insights gained from solving for the optimal policy function in this case, together with the example on the choice of technique in the short and long-run in our first strand of research, will provide a more complete understanding of the nature of the optimal policy function in the general RSS model.

The plan of the paper is as follows. After describing basic properties of the RSS model in Section 2, we show the equivalence of optimal and minimum value-loss programs (in Section 3). Through an extension of the principal results of dynamic programming (in Section 4), we establish monotonicity properties of the optimal policy function (OPF). This enables us to provide a fairly complete description of the nature of the optimal policy function in Section 5. This description, however, is less precise than in the RSS model with a linear felicity function, since the details of this description depend (as they should) on the concavity of the felicity function.

Under additional sufficient conditions on the "degree of concavity" of the felicity function, we refine the above description of the OPF in Section 6. Specifically, we narrow the range in which the OPF can lie, when the felicity function exhibits "limited" degree of concavity. In particular, if the degree of concavity of the felicity function is limited, we show that the OPF is identical to that obtained for the case of a linear felicity function, when the marginal rate of transformation between machines today and machines tomorrow (while maintaining full-employment of both labor and capital) is larger than unity.

2. Preliminaries

2.1. The model

We work with a special case of a two-sector model[1] in which the production of machines requires only labor. In the consumption-good sector, a single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-good sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation of McKenzie (1968, 2002), the collection of production plans (x, x'), the amount x' of machines at the end of the next period (tomorrow) from the amount x available at the end of the current period (today), is given by the *transition possibility set*:

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0 \ \text{ and } \ a(x' - (1-d)x) \leq 1\}$$

where $z \equiv (x' - (1-d)x)$ is the number of machines that are produced during the next period, and $z \geq 0$ and $az \leq 1$, respectively, formalize constraints on reversibility of investment and the use of labor. For any $(x, x') \in \Omega$, one can consider the amount y of the machines available for the production of the consumption good, leading to a correspondence $\Lambda : \Omega \to \mathbb{R}_+$ with $\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \ \textit{and} \ \ y \leq 1 - a(x' - (1-d)x)\}$. The parameters a and d will be

[1] This two-sector RSS model is a special case of the standard neoclassical two-sector model of Srinivasan (1964); Uzawa (1964).

assumed to satisfy the restriction:

$$\xi = \frac{1}{a} - (1 - d) > 0$$

This key parameter, ξ, measures the rate of transformation between machines today and machines tomorrow, while maintaining full-employment of both labor and capital.

Welfare is derived only from the consumption good and is represented by a function $w : \mathbb{R}_+ \to \mathbb{R}$, with y units of the consumption good yielding a welfare level $w(y)$. The function, w, is assumed to be continuous, increasing and strictly concave on $\mathbb{R}_+$ and twice continuously differentiable on $\mathbb{R}_{++}$, with $w'(y) > 0$ and $w''(y) < 0$ for all $y > 0$. A *reduced form utility function* $u : \Omega \to \mathbb{R}_+$ with $u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}$ indicates the maximum welfare level that can be obtained today, if one starts with x of machines today, and ends up with x' of machines tomorrow, where $(x, x') \in \Omega$.

An *economy* consists of (a, d, w), and the following concepts apply to it. A *feasible program* from x_0 is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_0$, and for all $t \in \mathbb{N}$, $(x(t), x(t + 1)) \in \Omega$ and $y(t) \in \Lambda(x(t), x(t + 1))$. A *program* from x_0 is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_0$, and for all $t \in \mathbb{N}$, $(x(t), x(t + 1)) \in \Omega$ and $y(t) = \max \Lambda(x(t), x(t + 1))$. A *program* $\{x(t), y(t)\}$ is a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t + 1)\}$, defined by $z(t + 1) = (x(t + 1) - (1 - d)x(t))$ and a *consumption sequence* $\{c(t + 1)\}$ defined by $c(t + 1) = y(t)$ for all $t \in \mathbb{N}$.[2] It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $M(x(0)) \equiv \max\{x(0), \bar{x}\}$, where $\bar{x} = (1/ad)$ is the *maximum sustainable capital stock*. A program $\{x(t), y(t)\}$ is called *stationary* if $(x(t), y(t)) = (x(t + 1), y(t + 1))$ for all $t \in \mathbb{N}$. For a stationary program $\{x(t), y(t)\}$, we have $x(t) \leq \bar{x}$ for all $t \in \mathbb{N}$.

A program $\{x^*(t), y^*(t)\}$ from x_0 is called *optimal* if:

$$\lim_{T\to\infty} \inf \sum_{t=0}^{T} [u(x(t), x(t + 1)) - u(x^*(t), x^*(t + 1))] \leq 0$$

for every program $\{x(t), y(t)\}$ from x_0. A *stationary optimal program* is a program that is stationary and optimal.

2.2. Full-employment programs

Our analysis in the following sections will be facilitated by focusing on *full-employment programs*, $\{x(t), y(t)\}$, which are programs satisfying: $y(t) + a[x(t + 1) - (1 - d)x(t)] = 1$ for all $t \in \mathbb{N}$. In this subsection, we note some basic properties relating to programs and full-employment programs, which will be useful in this connection.

It is fairly obvious that given any program, there is a full-employment program which has the *same* consumption sequence. Slightly more subtle is the property that higher initial stocks always allow programs which have *dominant* consumption sequences (at least as high in all periods, higher in at least one period). It is a consequence of this property that given any program which is *not* a full-employment program, there is a full-employment program (from the same initial stock) that has a dominant consumption sequence. We formally summarize these results in the following proposition[3].

Proposition 1.

(i) *If $\{x(t), y(t)\}$ is a program from x, then there is a full-employment program $\{x'(t), y'(t)\}$ from x, such that $y'(t) = y(t)$ for $t \geq 0$.*
(ii) *If $\{x(t), y(t)\}$ is a program from x, and $x' > x$ then there is a full-employment program $\{x'(t), y'(t)\}$ from x', such that $y'(t) \geq y(t)$ for all $t \in \mathbb{N}$, and $y'(t) > y(t)$ for some $t \in \mathbb{N}$.*
(iii) *If $\{x(t), y(t)\}$ is a program from x, which is not a full-employment program, then there is a full-employment program $\{x'(t), y'(t)\}$ from x,* such that *$y'(t) \geq y(t)$ for all $t \in \mathbb{N}$, and $y'(t) > y(t)$ for some $t \in \mathbb{N}$.*

[2] It is useful to note that for a program $\{x(t), y(t)\}$, since $y(t) = \max \Lambda(x(t), x(t + 1))$, the definition of u implies that $w(y(t)) = u(x(t), x(t + 1))$, since w is increasing.

[3] The proofs, being fairly straightforward, are omitted.

2.3. A golden-rule

The concept of a *golden-rule* plays a fundamental role in our analysis. Formally, we define a stock $\hat{x} \in \mathbb{R}_+$ as a *golden-rule stock* if $(\hat{x}, \hat{x}) \in \Omega$ and $u(\hat{x}, \hat{x}) \geq u(x, x')$ for all $(x, x') \in \Omega$ with $x' \geq x$. We record in the following result the existence and uniqueness of a golden-rule stock, and its "price support". For this purpose, we denote $1/(1 + ad)$ by $\hat{y}$, $a/(1 + ad)$ by $\hat{q}$, $w'(\hat{y})$ by m; then $\hat{y} > 0, \hat{q} > 0, m > 0$.

Proposition 2.

(i) *The pair* $(\hat{x}, \hat{p}) = (1/(1 + ad), m\hat{q})$ *satisfies* $(\hat{x}, \hat{x}) \in \Omega$, *and*:

$$u(\hat{x}, \hat{x}) \geq u(x, x') + \hat{p}x' - \hat{p}x \quad \textit{for all} \;\; (x, x') \in \Omega \tag{1}$$

(ii) $\hat{x}$ *is a unique golden-rule stock.*

Proof.

(i) It is straightforward to check that $(\hat{x}, \hat{x}) \in \Omega$ and $u(\hat{x}, \hat{x}) = w(\hat{y})$. Now, denote for $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$, $d\hat{p}(x - y)$ by $\alpha(x, x', y)$, and $(\hat{p}/a)\{1 - y - a[x' - (1 - d)x]\}$ by $\beta(x, x', y)$, and $w'(\hat{y})(y - \hat{y}) - (w(y) - w(\hat{y}))$ by $\gamma(x, x', y)$. Then, $\alpha(x, x', y) \geq 0$, $\beta(x, x', y) \geq 0$ and $\gamma(x, x', y) \geq 0$, the last inequality following from the concavity and continuity of w on $\mathbb{R}_+$, and its differentiability on $\mathbb{R}_{++}$. We can now do the following elementary computations:

$$\begin{aligned} y + \hat{q}x' - \hat{q}x &= y + \hat{q}[x' - (1 - d)x] - d\hat{q}x \\ &= \left[\frac{1}{1 + ad}\right] y + \hat{q}[x' - (1 - d)x] - \left[\frac{\alpha(x, x', y)}{m}\right] \\ &= \left[\frac{1}{1 + ad}\right] \{y + a[x' - (1 - d)x]\} - \left[\frac{\alpha(x, x', y)}{m}\right] \\ &= \hat{x}\{y + a[x' - (1 - d)x]\} - \left[\frac{\alpha(x, x', y)}{m}\right] \\ &= \hat{x} - \hat{x}\{1 - y - a[x' - (1 - d)x]\} - \left[\frac{\alpha(x, x', y)}{m}\right] \\ &= \hat{y} - \left[\frac{\beta(x, x', y)}{m}\right] - \left[\frac{\alpha(x, x', y)}{m}\right] \end{aligned} \tag{2}$$

Then, we have:

$$\begin{aligned} w(y) + \hat{p}x' - \hat{p}x &= [w(y) - w'(\hat{y})y] - [w(\hat{y}) - w'(\hat{y})\hat{y}] + w'(\hat{y})(y - \hat{y}) \\ &\quad + w(\hat{y}) + \hat{p}x' - \hat{p}x \\ &= my + \hat{p}x' - \hat{p}x - m\hat{y} + w(\hat{y}) - \gamma(x, x', y) \\ &= w(\hat{y}) - \alpha(x, x', y) - \beta(x, x', y) - \gamma(x, x', y) \end{aligned} \tag{3}$$

the last line of (3) following from (2). Clearly, (3) yields (1) from the definition of u.

(ii) If x is a golden-rule stock, then $(x, x) \in \Omega$ and $y \in \Lambda(x, x)$, with $y = \hat{y}$. Thus, using (3), we must have $\gamma(x, x, y) = \beta(x, x, y) = \alpha(x, x, y) = 0$. Clearly, $\alpha(x, x, y) = 0$ implies $x = y = \hat{x}$. □

We refer to $\hat{p}$, given by Proposition 2, as the *golden-rule price* and to the pair $(\hat{x}, \hat{p})$ as the *g* olden-rule. The golden-rule price provides a *price-support* to the golden-rule stock in the sense conveyed precisely by (1). The *v* on Neumann facet is the set of those $(x, x') \in \Omega$ for which equality holds in (1).

The *value-loss*(relative to the golden-rule) from operating at $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$ is:

$$\delta(x, x', y) = u(\hat{x}, \hat{x}) - [u(x, x') + \hat{p}x' - \hat{p}x] \tag{4}$$

Clearly, excess-capacity of capital and unemployment of labor are possible sources of value-loss. In addition, because w is strictly concave, a value-loss occurs whenever the consumption level, y, is different from the golden-rule consumption level, $\hat{y} = \hat{x}$. This can be formalized by splitting up the value-loss in (4) into three parts as follows (the splitting up already used in the proof of Proposition 2): $\delta(x, x', y) = \alpha(x, x', y) + \beta(x, x', y) + \gamma(x, x', y)$, where:

$$\left.\begin{aligned} \alpha(x, x', y) &= \hat{p}d(x - y) \\ \beta(x, x', y) &= \left(\frac{\hat{p}}{a}\right)(1 - y - a(x' - (1 - d)x)) \\ \gamma(x, x', y) &= w'(\hat{y})(y - \hat{y}) - (w(y) - w(\hat{y})) \end{aligned}\right\} \tag{5}$$

Let $\{x(t), y(t)\}$ be a program; then, $(x(t), x(t + 1)) \in \Omega$ and $w(y(t)) = u(x(t), x(t + 1))$ for $t \in \mathbb{N}$. Thus, using (4) and (5), we have for $t \in \mathbb{N}$:

$$\begin{aligned} u(\hat{x}, \hat{x}) &= u(x(t), x(t + 1)) + \hat{p}x(t + 1) - \hat{p}x(t) + \alpha(t) + \beta(t) + \gamma(t) \\ &\equiv u(x(t), x(t + 1)) + \hat{p}x(t + 1) - \hat{p}x(t) + \delta(t) \end{aligned} \tag{6}$$

where $\alpha(t) = \alpha(x(t), x(t + 1), y(t))$, $\beta(t) = \beta(x(t), x(t + 1), y(t))$, $\gamma(t) = \gamma(x(t), x(t + 1), y(t))$ and $\delta(t) = \delta(x(t), x(t + 1), y(t))$ for $t \in \mathbb{N}$. The program is called *good* if there is a real number G such that for all $T \in \mathbb{N}$, we have: $\sum_{t=0}^{T}[u(x(t), x(t + 1)) - u(\hat{x}, \hat{x})] \geq G$.

2.4. Long-run dynamics of good programs

Long-run optimal behavior can be described by studying the long-run dynamics of good programs. Every good program is seen to converge to the golden-rule stock, a *turnpike* property. This result is derived by using the price-support property in Proposition 2, and the full-employment property in Proposition 1.

Theorem 1.

(i) *There exists a good program* $\{x(t), y(t)\}$ *from every* $x_0 \in \mathbb{R}_+$.
(ii) *If* $\{x(t), y(t)\}$ *is a good program, then* $\sum_{t=0}^{\infty}\delta(t) < \infty$.
(iii) *If* $\{x(t), y(t)\}$ *is a good program, then* $y(t) \to \hat{y}$ *as* $t \to \infty$.
(iv) *If* $\{x(t), y(t)\}$ *is a good program, then* $x(t) \to \hat{x}$ *as* $t \to \infty$.
(v) *If* $\{x(t), y(t)\}$ *is a good program, then*:

$$\sum_{t=0}^{\infty}[u(x(t), x(t + 1)) - u(\hat{x}, \hat{x})] = \hat{p}x - \hat{p}\hat{x} - \sum_{t=0}^{\infty}\delta(t) \tag{US}$$

Proof.

(i) Define $y(0) = 0$ and $y(t + 1) = (1 - d)y(t) + d\hat{x}$ for $t \geq 0$. Then, $y(t)$ is monotonically non-decreasing, and converges to $\hat{x} = \hat{y}$ as $t \to \infty$. Define $z(t + 1) = d\hat{x}$ for $t \geq 0$. Given an arbitrary initial stock, x, define $x(0) = x$, and $x(t + 1) = (1 - d)x(t) + z(t + 1)$ for $t \geq 0$. Then, it is easy to check that $\{x(t), y(t)\}$ is a program from x. Given the definition of the sequence $\{y(t)\}$, we have $(y(t) - \hat{y}) = (1 - d)^t(y(0) - \hat{y})$ for $t \geq 0$. Thus, the sequence $\{\hat{y} - y(t)\}$ is summable, and $(\hat{y} - y(t))$ converges to zero. Thus, we can find $T \in \mathbb{N}$ such that $y(t) \geq \hat{y}/2$ for all $t \geq T$. Thus, we have:

$$w(y(t)) - w(\hat{y}) \geq w'\left(\frac{\hat{y}}{2}\right)(y(t) - \hat{y}) = w'\left(\frac{\hat{y}}{2}\right)(1 - d)^t(y(0) - \hat{y}) \quad \textit{for} \;\; t \geq T$$

This shows that $\{w(\hat{y}) - w(y(t))\}$ is summable, so $\{x(t), y(t)\}$ is a good program from x.

(ii) Using (6), we can write, for $T \in \mathbb{N}$:

$$\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = \hat{p}x(0) - \hat{p}x(T+1) - \sum_{t=0}^{T}\delta(t)$$

Since $\{x(t), y(t)\}$ is good, there is a real number G, such that $\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] \geq G$, for all $T \in \mathbb{N}$. So, $\sum_{t=0}^{T}\delta(t) \leq \hat{p}x(0) - G$ for all $T \in \mathbb{N}$, and consequently $\sum_{t=0}^{\infty}\delta(t) < \infty$.

(iii) Define $J = [0, Mx(0)]$, and a function, $H : J \to \mathbb{R}$ by:

$$H(y) = w'(\hat{y})(y - \hat{y}) - (w(y) - w(\hat{y}))$$

Then, by strict concavity of w on $\mathbb{R}_+$ and continuous differentiability of w on $\mathbb{R}_{++}$, we have:

$$H(y) > 0 \text{ for all } y \in J,\ y \neq \hat{y} \tag{7}$$

Now, given any $0 < \varepsilon < \hat{y}$, the set $J(\varepsilon) = \{y \in J : |y - \hat{y}| \geq \varepsilon\}$ is a non-empty and compact set, and H is a continuous function on it. Consequently, H attains a minimum value on $J(\varepsilon)$, call it $\rho(\varepsilon)$. Using (7), $\rho(\varepsilon) > 0$. Thus, given any $\varepsilon \in (0, \hat{y})$, there is $\rho(\varepsilon) > 0$, such that:

$$y \in J(\varepsilon) \text{ implies } H(y) \geq \rho(\varepsilon) > 0 \tag{8}$$

This is essentially a form of the well-known value-loss lemma.[4]

If $\{x(t), y(t)\}$ is a good program, then by (ii) we have $\delta(t) \to 0$ as $t \to \infty$, and therefore $\gamma(t) \to 0$ as $t \to \infty$. Since $\gamma(t) = H(y(t))$, it follows that $H(y(t)) \to 0$ as $t \to \infty$. Then, (8) implies that $y(t) \to \hat{y}$ as $t \to \infty$.

(iv) If $\{x(t), y(t)\}$ is a good program, then by (ii) we have $\delta(t) \to 0$ as $t \to \infty$, and therefore $\alpha(t) \to 0$ as $t \to \infty$. Thus, we must have $(x(t) - y(t)) \to 0$ as $t \to \infty$. But, since $y(t) \to \hat{y} = \hat{x}$, we must have $x(t) \to \hat{x}$ as $t \to \infty$.

(v) If $\{x(t), y(t)\}$ is a good program from x, then using (6), we can write for every positive integer T:

$$\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = \hat{p}x - \hat{p}x(T+1) - \sum_{t=0}^{T}\delta(t) \tag{9}$$

By (ii), $\sum_{t=0}^{\infty}\delta(t) < \infty$ and by (iv), $x(t) \to \hat{x}$ as $t \to \infty$. Thus, the right-hand side of (9) has a limit as $T \to \infty$, and so the left-hand side of (9) also has a limit as $T \to \infty$. Taking limits in (9), by letting $T \to \infty$, we obtain (US). □

3. Equivalence of optimality and value-loss minimization

In this section, we study the implications of the value-loss approach to dynamic optimization problems in the context of the RSS model. The focus of the study becomes the relation between optimality and value-loss minimization.

We start with the result that value-loss minimization is *s* ufficient for optimality, an observation due to Brock (1970). We then show that value-loss minimization is also *necessary* for optimality. These results become the key ingredients in developing the dynamic programming approach in Section 4, which in turn allows us to explicitly solve for the optimal policy function in Section 5.

3.1. Value-loss minimization implies optimality

Brock (1970) showed that value-loss minimization is *sufficient* for optimality; therefore, establishing the existence of a program which minimizes the sum of value-losses enables one to establish the existence of an optimal program. For easy reference, we state his result for our framework.

[4] For the original value-loss lemma, which was established in the context of a model with a different optimality criterion, see Radner (1961).

Proposition 3.

(i) *If* $\{x(t), y(t)\}$ *is a program from* $x \in \mathbb{R}_+$, *such that*:

$$\sum_{t=0}^{\infty} \delta(t) \le \sum_{t=0}^{\infty} \delta'(t) \tag{VLM}$$

for every program $\{x'(t), y'(t)\}$ *from x, then* $\{x(t), y(t)\}$ *is optimal from x.*

(ii) *If* $x \in \mathbb{R}_+$, *there is a program* $\{x(t), y(t)\}$ *from x, such that*:

$$\sum_{t=0}^{\infty} \delta(t) \le \sum_{t=0}^{\infty} \delta'(t)$$

for every program $\{x'(t), y'(t)\}$ *from x.*

(iii) *If* $x \in \mathbb{R}_+$, *there is an optimal program from x.*

3.2. Optimality implies value-loss minimization

In our framework, good programs exist from every initial stock. Consequently, optimal programs are necessarily good, so optimality implies value-loss minimization, since the relevant "transversality condition":

$$\lim_{t\to\infty} \hat{p}(x(t) - \hat{x}) = 0$$

is satisfied along an optimal program $\{x(t), y(t)\}$.

Proposition 4.

(i) *If* $\{x(t), y(t)\}$ *is an optimal program from* $x \in \mathbb{R}_+$, *then it must be good. Further*,

$$\lim_{t\to\infty} \hat{p}(x(t) - \hat{x}) = 0 \tag{10}$$

and:

$$\sum_{t=0}^{\infty} \delta(t) \le \sum_{t=0}^{\infty} \delta'(t) \tag{VLM}$$

for every program $\{x'(t), y'(t)\}$ *from x.*

(ii) *If* $\{x(t), y(t)\}$ *is an optimal program from* $x \in \mathbb{R}_+$, *then*:

$$\sum_{t=0}^{\infty} [u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = \hat{p}x - \hat{p}\hat{x} - \sum_{t=0}^{\infty} \delta(t) \tag{US}$$

Proof.

(i) If $\{x(t), y(t)\}$ is an optimal program from x, and $\{x'(t), y'(t)\}$ is *any* program from x, then using (6), we can write for every positive integer $T \in \mathbb{N}$,

$$\sum_{t=0}^{T-1} [u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] = \hat{p}x'(0) - \hat{p}x'(T) - \sum_{t=0}^{T-1} \delta'(t) \tag{11}$$

and:

$$\sum_{t=0}^{T-1} [u(x'(t), x'(t+1)) - u(x(t), x(t+1))] = \hat{p}x(T) - \hat{p}x'(T) - \sum_{t=0}^{T-1} \delta'(t) + \sum_{t=0}^{T-1} \delta(t) \tag{12}$$

By Theorem 1(i), there is a good program $\{\bar{x}(t), \bar{y}(t)\}$ from x, and $\sum_{t=0}^{\infty}\bar{\delta}(t) < \infty$. So, using this program in place of $\{x'(t), y'(t)\}$ in (12), together with the fact that $\{x(t), y(t)\}$ is an optimal program from x, we can infer that $\sum_{t=0}^{\infty}\delta(t) < \infty$, since $\bar{x}(t) \le M(x)$ for $t \ge 0$. Now, using $\{x(t), y(t)\}$ in place of $\{x'(t), y'(t)\}$ in (11), we see that $\{x(t), y(t)\}$ must be good, since $x(t) \le M(x)$ for $t \ge 0$. Now, (10) follows from Theorem 1(iv).

Since, we have verified that $\{x(t), y(t)\}$ is good, we have $\sum_{t=0}^{\infty}\delta(t) < \infty$ by Theorem 1(ii). If $\{x'(t), y'(t)\}$ is not good, then since $x'(t) \le M(x)$ for $t \ge 0$, we can use (11) to infer that $\sum_{t=0}^{T-1}\delta'(t) \to \infty$ as $T \to \infty$, so that (VLM) is trivially satisfied. If $\{x'(t), y'(t)\}$ is good, then $\sum_{t=0}^{\infty}\delta'(t) < \infty$ by Theorem 1(ii). Further, from Theorem 1(iv), we have $\lim_{t\to\infty}\hat{p}x(t)=\hat{p}\hat{x}$ and $\lim_{t\to\infty}\hat{p}x'(t) = \hat{p}\hat{x}$. Thus, we see that the right-hand side of (12) has a limit as $T \to \infty$. So, the left-hand side of (12) has a limit as $T \to \infty$, and this limit equals $[\sum_{t=0}^{\infty}\delta(t) - \sum_{t=0}^{\infty}\delta'(t)]$. Now, (VLM) follows from the optimality of $\{x(t), y(t)\}$.

(ii) If $\{x(t), y(t)\}$ is an optimal program from x, then using (i), it is a good program. So (US) follows from Theorem 1(v). □

A consequence of the above proposition is that one can develop a completely satisfactory theory of undiscounted dynamic programming, comparable to the theory available in the discounted case, a topic we address fully in the next section.

4. Dynamic programming

In this section, we develop the principal results of dynamic programming for our model. We first show how the optimal policy and value functions can be defined, and then we establish the connection between the two via the functional equation of dynamic programming.

4.1. Optimal policy and value functions

Our exposition is considerably simplified by noting that an optimal program from each initial stock is necessarily unique, in both the consumption level and the input level.

Proposition 5. *There is a unique optimal program $\{x(t), y(t)\}$ from each $x \in \mathbb{R}_+$.*

Proof. By Proposition 3, we know that there exists an optimal program from each $x \in \mathbb{R}_+$. We proceed now to show the uniqueness claim. First, we show that the consumption sequence is necessarily unique, using the strict concavity of the welfare function, w.

Suppose $\{x(t), y(t)\}$ and $\{x'(t), y'(t)\}$ are optimal programs from x, with $y(t) \ne y'(t)$ for some $t \ge 0$. Using Proposition 4(ii), we have:

$$\left.\begin{aligned} \sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] &= \hat{p}x - \hat{p}\hat{x} - \sum_{t=0}^{\infty}\delta(t) \\ \sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] &= \hat{p}x - \hat{p}\hat{x} - \sum_{t=0}^{\infty}\delta'(t) \end{aligned}\right\} \tag{13}$$

Also, by Proposition 4(i), we have $\sum_{t=0}^{\infty}\delta(t) = \sum_{t=0}^{\infty}\delta'(t)$, so that (13) yields:

$$\sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = \sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] \tag{14}$$

The sequence $\{x''(t), y''(t)\}$, defined by $x''(t) = \frac{1}{2}x(t) + \frac{1}{2}x'(t)$, $y''(t) = \frac{1}{2}y(t) + \frac{1}{2}y'(t)$ for all $t \ge 0$, is clearly a program from x, and we have:

$$\left.\begin{aligned} w(y''(t)) &\ge \frac{1}{2}w(y(t)) + \frac{1}{2}w(y'(t)) \quad \text{for all } t \ge 0 \\ w(y''(t)) &> \frac{1}{2}w(y(t)) + \frac{1}{2}w(y'(t)) \quad \text{for some } t \ge 0 \end{aligned}\right\} \tag{15}$$

by using the strict concavity of w, and the fact that $y(t) \neq y'(t)$ for some $t \geq 0$. Thus, there is $\varepsilon > 0$ and $N \in \mathbb{N}$, such that for all $T \geq N$,

$$\sum_{t=0}^{T}[w(y''(t)) - w(\hat{y})] \geq \frac{1}{2}\sum_{t=0}^{T}[w(y(t)) - w(\hat{y})] + \frac{1}{2}\sum_{t=0}^{T}[w(y'(t)) - w(\hat{y})] + \varepsilon$$

This implies that for all $T \geq N$,

$$\sum_{t=0}^{T}[u(x''(t), x''(t+1)) - u(\hat{x}, \hat{x})] \geq \frac{1}{2}\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]$$
$$+\frac{1}{2}\sum_{t=0}^{T}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] + \varepsilon \quad (16)$$

Taking the lim inf of both sides in (16), and noting that the right-hand side of (16) has a limit, we have:

$$\lim_{T\to\infty} \inf \sum_{t=0}^{T}[u(x''(t), x''(t+1)) - u(\hat{x}, \hat{x})] \geq \frac{1}{2}\sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]$$
$$+\frac{1}{2}\sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] + \varepsilon = \sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] + \varepsilon$$

the final equality following from (14). This contradicts the optimality of $\{x(t), y(t)\}$. Thus, the consumption sequence of any optimal program from x is necessarily unique.

The uniqueness of the input sequence now follows easily. For any optimal program $\{x(t), y(t)\}$ from x, we have by the full-employment property of Proposition 1,

$$y(t) + a[x(t+1) - (1-d)x(t)] = 1 \quad (17)$$

Since, the sequence $\{y(t)\}$ is determined uniquely by x, using (17) for $t = 0$ determines $x(1)$ uniquely, since $x(0) = x$. Now, using (17) repeatedly for $t = 1, 2, 3, ..$ determines the entire sequence $\{x(t)\}$ uniquely. □

In view of Proposition 5, we can define a function, $h : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$h(x) = x(1) \quad (18)$$

where $\{x(t), y(t)\}$ is the (unique) optimal program from x. The function, h, is the *optimal policy function*. It can be checked that h is continuous on $\mathbb{R}_+$.

In view of property (US) obtained in Proposition 4, we can define a v alue function, $V : \mathbb{R}_+ \to \mathbb{R}$, by:

$$V(x) \equiv \sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] \quad (19)$$

where $\{x(t), y(t)\}$ is the (unique) optimal program from x. It is useful to note that for every good program $\{x'(t), y'(t)\}$ from x, we have, by using (US) and (VLM):

$$V(x) \geq \sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] \quad (20)$$

4.2. The basic result on dynamic programming

We can now summarize the principal result on dynamic programming in our framework as follows.

Theorem 2. (i) *The value function V is a concave and strictly increasing function on $\mathbb{R}_+$ and continuous on $\mathbb{R}_{++}$, satisfying $V(\hat{x}) = 0$;* (ii) *V satisfies the functional equation of dynamic programming (FED) $V(x) =$*

$\max_{x' \in \Omega(x)}\{[u(x, x') - u(\hat{x}, \hat{x})] + V(x')\}$; (iii) $\{x(t), y(t)\}$ *is the optimal program from* $x(0)$ *if and only if for all* $t \in \mathbb{N}$, $V(x(t)) = [u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] + V(x(t+1))$.

Proof.

(i) The golden-rule program, defined by $(\hat{x}(t), \hat{y}(t)) = (\hat{x}, \hat{x})$ has zero value loss in each period, and is therefore optimal from $\hat{x}$, by Proposition 3. Thus, $V(\hat{x}) = 0$ by (19).

If $\{x(t), y(t)\}$ is an optimal program from x, and $x' > x$, then we can define: $x'(0) = x'$, $x'(t+1) = x'(t) + z(t+1)$ for $t \geq 0$ and $y'(t) = y(t)$ for $t \geq 0$, where $\{z(t+1)\}$ is the investment sequence associated with the program $\{x(t), y(t)\}$. Then, it is easy to check that $x'(t) > x(t)$ for all $t \geq 0$, and so $\{x'(t), y'(t)\}$ is a program from x'. Thus, V is non-decreasing on $\mathbb{R}_+$.

A slight refinement of this argument shows that, in fact, V is strictly increasing on $\mathbb{R}_+$. First, note that we must have $z(t+1) > 0$ for some $t \geq 0$. Otherwise, if $z(t+1) = 0$ for all $t \geq 0$, then $x(t) \to 0$ as $t \to \infty$, which would contradict the turnpike property of optimal programs. Let T be the first period for which $z(T+1) > 0$. If $x' > x$, we can define: $x'(0) = x'$, $x'(t+1) = x'(t) + z(t+1)$ for $t \neq T$, $x'(T+1) = x'(T) + z'(T+1)$, where $z'(T+1) = z(T+1) - \varepsilon$, where $0 < \varepsilon < z(T+1)$ and ε is sufficiently close to 0 so that $x'(T+1) > x(T+1)$. Finally, define $y'(t) = y(t)$ for $t \neq T$ and $y'(T) = y(T) + \varepsilon$. It is easy to check that $x'(t) > x(t)$ for all $t \geq 0$, and that $\{x'(t), y'(t)\}$ is a program from x'. Thus, V is strictly increasing on $\mathbb{R}_+$.

Since, Ω is convex, and u is concave on Ω, the value function is concave on $\mathbb{R}_+$. It follows that V is continuous on $\mathbb{R}_{++}$.

(ii) Let $\{x(t), y(t)\}$ be the optimal program from x. Then:

$$V(x) = [u(x, x(1)) - u(\hat{x}, \hat{x})] + \sum_{t=1}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]$$

Note that the sequence $\{x'(t), y'(t)\}$ defined by $(x'(t), y'(t)) = (x(t+1), y(t+1))$ for all $t \geq 0$ defines a program from $x(1)$. Since $\{x(t), y(t)\}$ is an optimal program from x, it is a good program from x (Proposition 4), and so $\{x'(t), y'(t)\}$ is a good program from $x(1)$. Thus, we have:

$$\sum_{t=1}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = \sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] \leq V(x(1))$$

the inequality following from (20). This yields the inequality:

$$V(x) \leq [u(x, x(1)) - u(\hat{x}, \hat{x})] + V(x(1)) \tag{21}$$

On the other hand, if $(x, x') \in \Omega$, then by using the optimal program $\{x'(t), y'(t)\}$ from x', we can define a program $\{x(t), y(t)\}$ from x by: $(x(0), y(0)) = (x, \max \Lambda(x, x'))$ and $(x(t), y(t)) = (x'(t-1), y'(t-1))$ for $t \geq 1$. Since, the program $\{x'(t), y'(t)\}$ from x' is good (by Proposition 4), so is the program $\{x(t), y(t)\}$ from x. Thus, by (19) and (20), we obtain:

$$\begin{aligned} V(x) &\geq [u(x, x') - u(\hat{x}, \hat{x})] + \sum_{t=1}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] \\ &= [u(x, x') - u(\hat{x}, \hat{x})] + \sum_{t=1}^{\infty}[u(x'(t-1), x'(t)) - u(\hat{x}, \hat{x})] \\ &= [u(x, x') - u(\hat{x}, \hat{x})] + \sum_{t=0}^{\infty}[u(x'(t), x'(t+1)) - u(\hat{x}, \hat{x})] = [u(x, x') - u(\hat{x}, \hat{x})] + V(x') \end{aligned} \tag{22}$$

Thus, if $\{x(t), y(t)\}$ is an optimal program from x, equality must hold in (21). There are two implications of this finding. First, we must have:

$$V(x(1)) = \sum_{t=1}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]$$

and therefore (by Proposition 5 and (19)) the sequence $\{x'(t), y'(t)\}$ defined by $(x'(t), y'(t)) = (x(t+1), y(t+1))$ for all $t \geq 0$ defines the optimal program from $x(1)$. Thus, we must have for each $t \geq 0$:

$$V(x(t)) = [u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] + V(x(t+1)) \tag{23}$$

Second, V satisfies the following *functional equation of dynamic programming*:

$$V(x) = \max_{(x,x') \in \Omega} \{[u(x, x') - u(\hat{x}, \hat{x})] + V(x')\} \tag{24}$$

(iii) We have already established one half of (iii) in (23) above. To establish the other half, let $\{x(t), y(t)\}$ be any program which satisfies for each $t \in \mathbb{N}$:

$$V(x(t)) = [u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] + V(x(t+1))$$

Then, for every $T \in \mathbb{N}$, we have:

$$V(x(0)) = \sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] + V(x(T+1)) \tag{25}$$

Since $x(t) \leq M(x(0))$ for all $t \in \mathbb{N}$ and V is increasing on X, defining $m = V(M(x(0)))$, we have $V(x(t)) \leq m$ for all $t \in \mathbb{N}$. On using this information in (25), we obtain:

$$\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] = V(x(0)) - V(x(T+1)) \geq V(x(0)) - m$$

allowing us to conclude that the program $\{x(t), y(t)\}$ must be good, and that therefore $x(t) \to \hat{x}$ as $t \to \infty$. Thus, by the continuity of V on $\mathbb{R}_{++}$, we obtain $V(x(t)) \to V(\hat{x}) = 0$ as $t \to \infty$. Again, using this information in (25), we see that $\{\lim_{T\to\infty}\sum_{t=0}^{T}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]\}$ exists and is finite, and:

$$V(x) = \sum_{t=0}^{\infty}[u(x(t), x(t+1)) - u(\hat{x}, \hat{x})]$$

Thus, by (19) and Proposition 5, $\{x(t), y(t)\}$ is the optimal program from $x(0)$. □

5. The nature of the optimal policy function

In this section, we use the theory developed in the previous three sections to describe the nature of the optimal policy function (OPF). Because the welfare function, w, is strictly concave, the policy function can be expected to depend on the degree of concavity of the function, w. Thus, in general, the best we can hope for is to describe the ranges in which the optimal policy function will lie. This is done in the second subsection. Of paramount importance for the analysis carried out there is a result dealing with the slope of the value function. We highlight this result by devoting the first subsection to it.

5.1. Slope of the value function

Since any program starting from a point in $X \equiv [0, 1/ad]$ remains in X, without further mention we will use X as the state space, and confine our solution of the optimal policy function to this domain.

Lemma 1. *The value function, V, satisfies the following properties*:

(i) *for all* $x > \hat{x}$, *we have*:

$$V(x) - V(\hat{x}) \le \hat{p}(x - \hat{x}) \tag{26}$$

(ii) $V'_+(\hat{x}) \le \hat{p} < ma$.

(iii) *for all* $x' > x \ge \hat{x}$, *we have*:

$$V(x') - V(x) \le V'_+(x)(x' - x) \le V'_+(\hat{x})(x' - x) \le \hat{p}(x' - x) < ma(x' - x) \tag{27}$$

Proof.

(i) This follows from (19) and (US).

(ii) Using (i), and letting $x \downarrow \hat{x}$, one gets $V'_+(\hat{x}) \le \hat{p}$. Using the definition of $\hat{p}$, we have $\hat{p} < ma$.

(iii) The first inequality of (27) follows from the concavity of V on $\mathbb{R}_+$, and $x > 0$. The second inequality follows from the concavity of V and $x \ge \hat{x} > 0$. The third and fourth inequalities follow from (ii). □

5.2. General qualitative properties of the OPF

In order to describe the nature of the policy function, it is convenient to subdivide X into three ranges of stocks:

$$k \equiv \frac{\hat{x}}{1-d}, \quad A = [0, \hat{x}], \quad B = (\hat{x}, 1), \quad C = \left[1, \frac{1}{ad}\right] \tag{28}$$

We refer to A as the range of "low stocks", B as the range of "intermediate stocks" and C as the range of "high stocks". For these ranges of stocks, one can establish the following characterization of the optimal policy function.

One way of viewing the result is the implication contained in it that when the stock is below(above) the golden-rule stock, consumption must be below(above) the golden-rule consumption.[5]

Proposition 6. *The optimal policy function*, $h : X \to \mathbb{R}_+$*satisfies*:

$$h(x) \in \begin{cases} \left\{\dfrac{1}{a} - \xi x\right\} & \textit{for all } x \in A \\ \left[\dfrac{1}{a} - \xi x, \hat{x} + (1-d)(x - \hat{x})\right] & \textit{for all } x \in B \\ [(1-d)x, \hat{x} + (1-d)(x - \hat{x})] & \textit{for all } x \in C \end{cases} \tag{OPF}$$

Proof. We start by analyzing the case in which $x \in B \cup C$. In this case, we claim that:

$$h(x) \le \hat{x} + (1-d)(x - \hat{x}) \tag{29}$$

Suppose (29) is violated for some $x \in B \cup C$. Denote the right-hand side of (29) by $\bar{x}$. Note that $(x, \bar{x}) \in \Omega$, $x > \hat{x}$, $h(x) > \bar{x}$ (by hypothesis) $> \hat{x}$ and $\max \Lambda(x, \bar{x}) = \bar{y} = \hat{y}$. Further, $u(x, h(x)) = w(y)$, where $y = 1 - a[h(x) - (1-d)x]$

[5] This feature is emphasized in the period-by-period condition of Brock and Majumdar (1988), characterizing optimality of competitive programs.

by Proposition 1. Thus, we have:

$$\begin{aligned} V(x) &\geq u(x,\bar{x}) - u(\hat{x},\hat{x}) + V(\bar{x}) \\ &= w(\hat{y}) + V(\bar{x}) - V(h(x)) + V(h(x)) - u(\hat{x},\hat{x}) \\ &= w(\hat{y}) - u(x,h(x)) + V(\bar{x}) - V(h(x)) + u(x,h(x)) - u(\hat{x},\hat{x}) + V(h(x)) \\ &\geq -ma(\bar{x}-h(x)) + V(\bar{x}) - V(h(x)) + V(x) \\ &\quad > -ma(\bar{x}-h(x)) - ma(h(x)-\bar{x}) + V(x) = V(x) \end{aligned} \tag{30}$$

a contradiction. Note that the last inequality in (30) follows from (27), since $h(x) > \bar{x} > \hat{x}$, while the previous inequality follows from the concavity and continuous differentiability of w on $\mathbb{R}_{++}$. Thus, (29) is established.

If $x \in C$, then since $h(x) \geq (1-d)x$ by feasibility, (29) implies that $h(x) \in [(1-d)x, \hat{x} + (1-d)(x-\hat{x})]$, as claimed in (OPF).

Next, we consider the case in which $x \in A$, with $x > 0$. Denote $[(1/a) - \xi x]$ by $\bar{x}$; note that $\bar{x} \geq \hat{x}$. Suppose, contrary to the proposition, that there is some $x \in A$, with $x > 0$ and $h(x) \neq \bar{x}$. If $h(x) < \bar{x}$, then for $y \in \Lambda(x, h(x))$, we have $a[h(x) - (1-d)x] + y < a[\bar{x} - (1-d)x] + x = 1$, so labor is not fully employed, a contradiction to Proposition 1. If $h(x) > \bar{x}$, then $h(x) > \bar{x} \geq \hat{x}$, and so, we get $V(h(x)) - V(\bar{x}) < ma(h(x) - \bar{x})$ by Lemma 1. Using this, we can write:

$$\begin{aligned} V(x) &= u(x,h(x)) - u(\hat{x},\hat{x}) + V(h(x)) \\ &= w(1 - a[h(x) - (1-d)x]) + [V(h(x)) - V(\bar{x})] + V(\bar{x}) - u(\hat{x},\hat{x}) \\ &\quad < w(1 - a[h(x) - (1-d)x]) - w(1 - a[\bar{x} - (1-d)x]) \\ &\quad + w(1 - a[\bar{x} - (1-d)x]) + ma(h(x) - \bar{x}) + V(\bar{x}) - u(\hat{x},\hat{x}) \\ &\leq w'(x)a(\bar{x} - h(x)) + w(x) + ma(h(x) - \bar{x}) + V(\bar{x}) - u(\hat{x},\hat{x}) \\ &\leq w'(\hat{y})a(\bar{x} - h(x)) + ma(h(x) - \bar{x}) + w(x) + V(\bar{x}) - u(\hat{x},\hat{x}) \\ &\leq w(x) - w(\hat{x}) + V(\bar{x}) \leq V(x) \end{aligned} \tag{31}$$

a contradiction. Note that the strict inequality in (31) follows from Lemma 1, and $h(x) > \bar{x} \geq \hat{x}$, and the next two weak inequalities follow from concavity and differentiability of w on $\mathbb{R}_{++}$, and the fact that $h(x) > \bar{x}$ while $0 < x \leq \hat{x} = \hat{y}$. This establishes that $h(x) = [(1/a) - \xi x]$ for $0 < x \leq \hat{x}$, and $h(0) = (1/a)$ follows by continuity of h. Thus, $h(x) = [(1/a) - \xi x]$ for $x \in A$, as claimed in (OPF).

Next, we consider the case in which $x \in B$. Denote $[(1/a) - \xi x]$ by $\bar{x}$. Suppose, contrary to the proposition, that there is some $x \in B$, with $h(x) < \bar{x}$. Then, for $y \in \Lambda(x, h(x))$, we have $a[h(x) - (1-d)x] + y < a[\bar{x} - (1-d)x] + x = 1$, so labor is not fully employed, a contradiction to Proposition 1. Thus, we must have $h(x) \geq [(1/a) - \xi x]$ for all $x \in B$. Also, we have shown that (29) holds for all $x \in B$. Thus, we have $h(x) \in [(1/a) - \xi x, \hat{x} + (1-d)(x-\hat{x})]$ for all $x \in B$, as claimed in (OPF). □

5.3. Monotone properties of the OPF

We can describe the nature of the optimal policy function somewhat more precisely by deriving certain monotone properties of it for capital stocks exceeding the golden-rule stock.[6] We describe two results in this connection, and then discuss their implications. Note, in this connection, that we already know exactly the policy function for capital stocks below the golden-rule stock (the "low stocks"), and so our efforts may be seen as trying to narrow the range of possible values of the optimal policy function, for capital stocks which are "high" or which lie in the "intermediate range" , compared to Proposition 6.

Proposition 7. *For $x \in C$, the optimal policy function is monotone non-decreasing.*

[6] In the literature, monotone properties of optimal policy correspondences have been most effectively used to study optimal behavior on non-convex feasible sets. See, for example, Dechert and Nishimura (1983); Majumdar and Nermuth (1982); Mitra and Ray (1984).

Proof. Let $x(0)$ and $x'(0)$ belong to C, with $x'(0) > x(0)$. Denote $h(x(0))$ by $x(1)$ and $h(x'(0))$ by $x'(1)$. We want to show that $x'(1) \geq x(1)$. Suppose, on the contrary,

$$x'(1) < x(1) \tag{32}$$

Following Dechert and Nishimura (1983), we now construct two alternative programs. The first goes from $x(0)$ to $x'(1)$ and then follows the optimal program from $x'(1)$; the second goes from $x'(0)$ to $x(1)$ and then follows the optimal program from $x(1)$. A crucial aspect of this technique in the current context (given the various production constraints) is that one be able to go from $x(0)$ to $x'(1)$, and from $x'(0)$ to $x(1)$. That is, one needs to show that $(x(0), x'(1)) \in \Omega$ and $(x'(0), x(1)) \in \Omega$.

We first check that $(x(0), x'(1)) \in \Omega$. Note that the irreversibility constraint is satisfied, since $x'(1) \geq (1-d)x'(0) > (1-d)x(0)$. Further, using (32), we have:

$$a[x'(1) - (1-d)x(0)] < a[x(1) - (1-d)x(0)] \leq 1$$

so that the labor constraint is satisfied if:

$$\bar{y} = 1 - a[x'(1) - (1-d)x(0)] > 0$$

is the amount of labor devoted to the production of the consumption good. Finally, the capital constraint is satisfied, since $\bar{y} \leq 1 \leq x(0)$, since $x(0) \in C$.

Next, we check that $(x'(0), x(1)) \in \Omega$. Note that the irreversibility constraint is satisfied, since (by using (32)), we have $x(1) > x'(1) \geq (1-d)x'(0)$. Further, since $x'(0) > x(0)$,

$$a[x(1) - (1-d)x'(0)] < a[x(1) - (1-d)x(0)] \leq 1$$

so that the labor constraint is satisfied if:

$$\tilde{y} = 1 - a[x(1) - (1-d)x'(0)] > 0$$

is the amount of labor devoted to the production of the consumption good. Finally, the capital constraint is satisfied, since:

$$\begin{aligned} \tilde{y} &= 1 - ax(1) + a(1-d)x'(0) \\ &< 1 - ax'(1) + a(1-d)x'(0) \\ &= y'(0) \leq x'(0) \end{aligned}$$

the inequality following from (32).

We can now use the technique of Dechert and Nishimura (1983), and we write out the steps to keep our exposition self-contained. First, from the definition of the optimal policy function, we have:

$$V(x(0)) = w(y(0)) - w(\hat{y}) + V(x(1)); \quad V(x'(0)) = w(y'(0)) - w(\hat{y}) + V(x'(1)) \tag{33}$$

Second, by the principle of optimality, we have:

$$V(x(0)) \geq w(\bar{y}) - w(\hat{y}) + V(x'(1)); \quad V(x'(0)) \geq w(\tilde{y}) - w(\hat{y}) + V(x(1)) \tag{34}$$

Clearly, (33) and (34) yield the inequality:

$$w(y(0)) + w(y'(0)) \geq w(\bar{y}) + w(\tilde{y}) \tag{35}$$

Next, note that:

$$\bar{y} + \tilde{y} = 1 - a[x'(1) - (1-d)x(0)] + 1 - a[x(1) - (1-d)x'(0)] = y(0) + y'(0) \tag{36}$$

and, using $x'(0) > x(0)$, and (32), we have:

$$y(0) = 1 - ax(1) + a(1-d)x(0) < 1 - ax(1) + a(1-d)x'(0) = \tilde{y} < 1 - ax'(1) + a(1-d)x'(0) = y'(0) \tag{37}$$

Using (37), there is $\theta \in (0, 1)$, such that $\tilde{y} = \theta y(0) + (1-\theta)y'(0)$, and using this in (36), we have $\bar{y} = \theta y'(0) + (1-\theta)y(0)$. Then, by the strict concavity of w, we get:

$$w(\tilde{y}) > \theta w(y(0)) + (1-\theta)w(y'(0)); \quad w(\bar{y}) > \theta w(y'(0)) + (1-\theta)w(y(0)) \tag{38}$$

Adding the inequalities in (38), we contradict (35). This establishes the proposition. □

It will be noted that the only place we make use of the fact that $x(0) \in C$ is in checking that $\bar{y} \leq x(0)$. Thus, if one can verify that this inequality holds, the optimal policy function can be shown to be monotone non-decreasing on an extended domain. The next result exploits this idea, and establishes a "local" monotonicity property of the OPF.

Proposition 8. *Suppose $x^* \in B$ and $h(x^*) > (1/a) - \xi x^*$, then there is $\varepsilon > 0$, such that $N(\varepsilon) \equiv (x^* - \varepsilon, x^* + \varepsilon) \subset B$, and the optimal policy function is monotone non-decreasing on $N(\varepsilon)$.*

Proof. Denote $\{1 - a[h(x^*) - (1-d)x^*]\}$ by y^*. Then, we have:

$$\begin{aligned} y^* &= 1 - ah(x^*) + a(1-d)x^* \\ &< 1 - a\left[\frac{1}{a} - \xi x^*\right] + a(1-d)x^* \\ &= a\xi x^* + a(1-d)x^* \\ &= a\left[\frac{1}{a} - (1-d)\right] x^* + a(1-d)x^* \\ &= x^* \end{aligned}$$

Denote $(x^* - y^*)$ by μ. Then, $\mu > 0$, and by continuity of h, we can find $\varepsilon > 0$, such that $N(\varepsilon) \equiv (x^* - \varepsilon, x^* + \varepsilon) \subset B$, and $[1 + a(1-d)]\varepsilon \leq (\mu/2)$, and $|h(x) - h(x^*)| < (\mu/2a)$ for all $x \in N(\varepsilon)$.

Now, let $x(0)$ and $x'(0)$ belong to $N(\varepsilon)$, with $x'(0) > x(0)$. We have to show that $x'(1) \equiv h(x'(0)) \geq h(x(0)) \equiv x(1)$. Suppose, on the contrary,

$$x'(1) < x(1) \tag{39}$$

Define $\bar{y}$ and $\tilde{y}$ as in the proof of Proposition 7. Then, one can arrive at a contradiction by following exactly the proof of Proposition 7, if one can show that $\bar{y} \leq x(0)$. To this end, note that:

$$\begin{aligned} \bar{y} = &\ 1 - ax'(1) + a(1-d)x(0) \\ &\leq 1 - ah(x^*) + \left(\frac{\mu}{2}\right) + a(1-d)x^* + a(1-d)\varepsilon \\ &= y^* + \left(\frac{\mu}{2}\right) + a(1-d)\varepsilon \\ &= x^* - \left(\frac{\mu}{2}\right) + a(1-d)\varepsilon \\ &= x^* - \varepsilon + [1 + a(1-d)]\varepsilon - \left(\frac{\mu}{2}\right) \\ &\leq x^* - \varepsilon < x(0) \end{aligned}$$

This completes the proof of the proposition. □

As an application of the monotonicity property, we can say a bit more about the nature of the optimal policy function on the domain $(\hat{x}, 1]$. It is convenient to define the function H by:

$$H(x) = \left[\frac{1}{a} - \xi x\right] \quad \text{for all } x \in [0, 1]$$

Corollary 1. *Suppose there is some $\tilde{x} \in (\hat{x}, 1]$ such that $h(\tilde{x}) = H(\tilde{x})$. Then, $h(x) = H(x)$ for all $x \in [\hat{x}, \tilde{x}]$.*

Proof. If not, there is some $x' \in [\hat{x}, \tilde{x}]$ such that $h(x') > H(x')$ by using Proposition 6. Let $x'' = \inf\{x \in [x', \tilde{x}] : h(x) = H(x)\}$. Since $h(\tilde{x}) = H(\tilde{x})$, this is well defined, and by continuity of h and H, we have $x'' > x'$, $h(x'') = H(x'')$ and $h(x) > H(x)$ for all $x \in (x', x'')$, by Proposition 6. Then by Proposition 8, we have $D_+h(x) \geq 0$ for all $x \in (x', x'')$. Thus, using the continuity of h, we have $h(x'') \geq h(x')$ (see Royden, 1988, Proposition 2, p. 99). But since $H(x'') = h(x'')$ and $h(x') > H(x')$, this implies that $H(x'') > H(x')$, which contradicts the fact that H is decreasing on $[0, 1]$. □

5.4. The OPF when $\xi = 1$

Using the monotone properties of the previous section, we can narrow down the range of possible values that the OPF can take, by analyzing in more detail the nature of the OPF, given different values of the technologically determined parameter, ξ. In this subsection, we start this detailed analysis by considering the case in which $\xi = 1$.

Proposition 9. *If $\xi = 1$, then the OPF, $h : X \to X$, must satisfy $h(x) \geq \hat{x}$ for all $x \in X$.*

Proof. We claim that $h(x) \geq \hat{x}$ for all $x \in B$. Suppose this were not true. Then, using Proposition 6, there is some $x \in B$, such that $h(x) \in [H(x), \hat{x})$, where:

$$H(x) = \left[\frac{1}{a} - \xi x\right] \quad \text{for all } x \in [0, 1]$$

There are two possibilities to consider: (i) $h(x) = H(x)$ and (ii) $h(x) > H(x)$.

In case (i), we have $h(x) = [(1/a) - \xi x] < [(1/a) - \xi \hat{x}] = \hat{x}$, and so $h(x) \in A$. Thus, we must have, by Proposition 6,

$$\begin{aligned} h(h(x)) &= H(h(x)) = H(H(x)) = \left[\frac{1}{a} - \xi H(x)\right] \\ &= \left[\frac{1}{a} - H(x)\right] \\ &= \left[\frac{1}{a} - \left\{\frac{1}{a} - x\right\}\right] \\ &= x \end{aligned}$$

using the fact that $\xi = 1$. Thus, $(x, H(x), x, H(x), \ldots.)$ is optimal from x. But this violates the fact that the sequence of stocks along every optimal program must converge to the golden-rule stock.

In case (ii), we have $h(x) \in (H(x), \hat{x})$. Thus, we can apply Proposition 8 and the continuity of h to infer that there is some $\tilde{x} \in (\hat{x}, x)$ such that $h(\tilde{x}) = H(\tilde{x})$. Then, we have $\tilde{x} \in B$, satisfying $h(\tilde{x}) = H(\tilde{x})$. So, applying the analysis of case (i) to $\tilde{x} \in B$, we again arrive at a contradiction. This establishes our claim that $h(x) \geq \hat{x}$ for all $x \in B$. Using Proposition 6 and the continuity of h, we therefore have:

$$h(x) \geq \hat{x} \quad \text{for all } x \in [0, 1]$$

Using Proposition 7, we then have $h(x) \geq \hat{x}$ for all $x \in [1, 1/ad]$, establishing the pro- position. □

5.5. The OPF when $\xi > 1$

We now analyze the OPF when the technologically determined parameter $\xi > 1$. We show that the conclusion obtained in Proposition 9 is also valid in this case (Fig. 1). The conclusion, however, has to be obtained by a somewhat different argument.

Let us define a function $v : \Omega \to \mathbb{R}$ by:

$$v(x, x') = \max\{my : y \in \Lambda(x, x')\}$$

where $m = w'(\hat{y})$ as in Section 2.3. Then, for $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$, we have:

$$y + \hat{q}x' - \hat{q}x = \hat{y} - \left[\frac{\beta(x, x', y)}{m}\right] - \left[\frac{\alpha(x, x', y)}{m}\right]$$

using (2), where (as in Section 2.3) $d\hat{p}(x - y) \equiv \alpha(x, x', y)$, and $(\hat{p}/a)\{1 - y - a[x' - (1 - d)x]\} \equiv \beta(x, x', y)$. Thus, we have:

$$my + \hat{p}x' - \hat{p}x = m\hat{y} - \beta(x, x', y) - \alpha(x, x', y) \tag{40}$$

where $\hat{p} = m\hat{q}$ (as in Section 2.3).

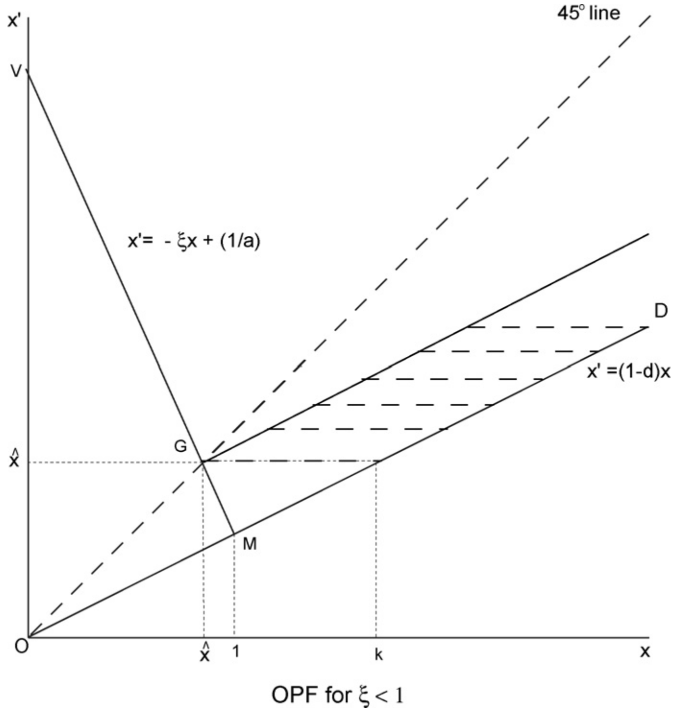

Fig. 1. Optimal policy function.

Let $\{x(t), y(t)\}$ be a program; then $(x(t), x(t+1)) \in \Omega$ and $w(y(t)) = u(x(t), x(t+1))$ for $t \in \mathbb{N}$. Thus, we have $v(\hat{x}, \hat{x}) = m\hat{y}$ and $v(x(t), x(t+1)) = my(t)$, and using (40), we obtain for $t \in \mathbb{N}$:

$$\begin{aligned} m\hat{y} &= v(\hat{x}, \hat{x}) = v(x(t), x(t+1)) + \hat{p}x(t+1) - \hat{p}x(t) + \alpha(t) + \beta(t) \\ &= my(t) + \hat{p}x(t+1) - \hat{p}x(t) + \alpha(t) + \beta(t) \end{aligned} \tag{41}$$

where $\alpha(t) = \alpha(x(t), x(t+1), y(t))$, $\beta(t) = \beta(x(t), x(t+1), y(t))$ for $t \in \mathbb{N}$.

We can eliminate the variable $my(t) = v(x(t), x(t+1))$ from (41), by using the fact that:

$$d\hat{p}(x(t) - y(t)) \equiv \alpha(x(t), x(t+1), y(t)) = \alpha(t)$$

to obtain for all $t \in \mathbb{N}$:

$$mx(t+1) = \frac{m}{a} - \xi mx(t) + A(t) - B(t)$$

where $\xi = [(1/a) - (1-d)]$, and $A(t) = (1/ad\hat{q})\alpha(t)$, $B(t) = (1/\hat{q})\beta(t)$ for $t \in \mathbb{N}$. On measuring capital stocks relative to the golden-rule stock, $X(t) = (x(t) - \hat{x})$, and on noting that $\hat{x} = (1/a) - \xi\hat{x}$, we can obtain the basic dynamic equation:

$$mX(t+1) = (-\xi)mX(t) + A(t) - B(t) \quad \textit{for all } t \in \mathbb{N} \tag{42}$$

This equation can be used to obtain the following useful result.

Lemma 2. *For any full-employment program* $\{x(t), y(t)\}$,

$$\sum_{t=0}^{T} \left[\frac{\alpha(t)}{(-\xi)^t}\right] = ad\xi \left\{\hat{p}X(0) - \hat{p}\left[\frac{X(T+1)}{(-\xi)^{T+1}}\right]\right\} \quad \textit{for all } T > 1 \tag{43}$$

Proof. If $\{x(t), y(t)\}$ is a full-employment program, then $B(t) = 0$ for all $t \in \mathbb{N}$. Thus, we can use (42) to write:

$$\left[\frac{A(t)}{\xi^t}\right] = \left[\frac{mX(t+1)}{\xi^t}\right] + \xi\left[\frac{mX(t)}{\xi^t}\right] \text{ for } t \in \mathbb{N} \tag{44}$$

Given any positive integer $T \in \mathbb{N}$, we now sum (44) from $t = 0$ to $t = T$, except that for t odd, we use the equation in (44) with a negative sign. This yields:

$$\sum_{t=0}^{T} \left[\frac{A(t)}{(-\xi)^t}\right] = \xi \left\{mX(0) - \left[\frac{mX(T+1)}{(-\xi)^{T+1}}\right]\right\} \tag{45}$$

Using the fact that $A(t) = [\alpha(t)/\hat{q}ad]$ for $t \in \mathbb{N}$, and $\hat{p} = m\hat{q}$, (45) yields (43). □

Proposition 10. *If* $\xi > 1$, *then the OPF*, $h : X \to X$ *must satisfy* $h(x) \geq \hat{x}$ *for all* $x \in X$.

Proof. We claim that $h(x) \geq \hat{x}$ for all $x \in B$. Suppose, on the contrary, that there is some $x \in B$ such that $h(x) < \hat{x}$. Then, by Proposition 6, we have $h(x) \in [H(x), \hat{x})$, where

$$H(x) = \frac{1}{a} - \xi x \quad \text{for all } x \in [0, 1]$$

Denoting by $\{x'(t)\}$ the optimal program from x, we have $x'(1) \in [H(x), \hat{x})$.

Define $\{\bar{x}(t)\}$ by $\bar{x}(0) = x$, and $\bar{x}(t) = \hat{x}$ for $t \geq 0$. Then, noting that $(\bar{x}(t), \bar{x}(t+1)) \in \Omega$ for all $t \in \mathbb{N}$, we can define $\{\bar{y}(t)\}$ by $\bar{y}(t) = 1 - a[\bar{x}(t+1) - (1-d)\bar{x}(t)]$ for all $t \in \mathbb{N}$. Then, $\{\bar{x}(t), \bar{y}(t)\}$ is a full-employment program from x. Thus, we have $\bar{\beta}(t) = 0$ for $t \in \mathbb{N}$, and since $\bar{x}(t) = \hat{x}$ for $t \geq 1$, we also have $\bar{X}(t) = 0$ and $\bar{\alpha}(t) = 0$ for $t \geq 1$. Thus, using Lemma 2, we obtain:

$$\sum_{t=0}^{\infty}[\bar{\alpha}(t) + \bar{\beta}(t)] = \bar{\alpha}(0) = \sum_{t=0}^{\infty}\left[\frac{\bar{\alpha}(t)}{(-\xi)^t}\right] = ad\xi\hat{p}\bar{X}(0) \tag{46}$$

Now, since $\{x'(t), y'(t)\}$ is an optimal program from x, it is a full-employment program from x, and using Lemma 2 and $\xi > 1$, we obtain:

$$\sum_{t=0}^{\infty}[\alpha'(t) + \beta'(t)] = \sum_{t=0}^{\infty}\alpha'(t) \geq \sum_{t=0}^{\infty}\left(\frac{\alpha'(t)}{\xi^t}\right) \geq \sum_{t=0}^{\infty}\left[\frac{\alpha'(t)}{(-\xi)^t}\right] = ad\xi\hat{p}X(0) \tag{47}$$

Since $x'(1) < \hat{x}$, we have:

$$y'(1) = 1 - a[x'(1) - (1-d)x] > 1 - a[\hat{x} - (1-d)x] = \bar{y}(1)$$

and since $x > \hat{x}$, we have:

$$\bar{y}(1) = 1 - a[\hat{x} - (1-d)x] > 1 - a[\hat{x} - (1-d)\hat{x}] = \hat{y}$$

Thus, we have:

$$y'(1) > \bar{y}(1) > \hat{y} \tag{48}$$

The function $\gamma(x, x', y) = w'(\hat{y})(y - \hat{y}) - (w(y) - w(\hat{y}))$ is independent of $(x, x') \in \Omega$, and $\partial\gamma(x, x', y)/\partial y = w'(\hat{y}) - w'(y) > 0$ for all $y > \hat{y}$, since w is strictly concave in y. Thus, (48) implies that:

$$\gamma'(0) \equiv \gamma(x, x'(1), y'(1)) > \gamma(x, \bar{x}(1), \bar{y}(1)) \equiv \bar{\gamma}(0) \tag{49}$$

Since, $\bar{x}(t) = \hat{x}$ for all $t \geq 1$, we have $\bar{y}(t) = \hat{y}$ for all $t \geq 1$, and so $\bar{\gamma}(t) = 0$ for $t \geq 1$. Also, $\gamma'(t) \geq 0$ for $t \geq 1$. Thus, (49) yields:

$$\sum_{t=0}^{\infty}\gamma'(t) > \sum_{t=0}^{\infty}\bar{\gamma}(t)$$

Combining this with (46) and (47), we obtain:

$$\sum_{t=0}^{\infty}\delta'(t) > \sum_{t=0}^{\infty}\bar{\delta}(t)$$

which contradicts the value-minimization property of optimal programs. This establishes our claim that $h(x) \geq \hat{x}$ for all $x \in B$. Using Proposition 6 and the continuity of h, we therefore have:

$$h(x) \geq \hat{x} \quad \text{for all } x \in [0, 1]$$

Using Proposition 7, we then have $h(x) \geq \hat{x}$ for all $x \in [1, 1/ad]$, establishing the pro- position. □

5.6. The nature of optimal programs when $\xi \geq 1$

The analysis of the previous two subsections allow us to describe the nature of optimal programs very precisely. This can be formally stated in the following result.

Theorem 3. *Suppose$\xi \geq 1$. Let $\{x(t)\}$ be the optimal program from $x(0) \in X$.*

(i) *If $x(0) = \hat{x}$, then $x(t) = \hat{x}$ for $t \geq 0$.*
(ii) *If $x(0) > \hat{x}$, then $x(t)$ monotonically decreases to $\hat{x}$ as $t \uparrow \infty$.*
(iii) *If $x(0) < \hat{x}$, then $x(1) > \hat{x}$, and for $t \geq 1$, $x(t)$ monotonically decreases to $\hat{x}$ as $t \uparrow \infty$.*

Proof. Clearly, (i) follows from Proposition 6. It is enough to show (ii), since (iii) follows from (ii) and Proposition 6.

For $\xi \geq 1$, Propositions 9 and 10 show that $h(x) \geq \hat{x}$ for all $x \in X$. Also, by Proposition 6, we know that $h(x) \leq \hat{x} + (1-d)(x-\hat{x}) = d\hat{x} + (1-d)x \leq x$ for $x \geq \hat{x}$. Thus, we must have $x(t) \geq x(t+1) \geq \hat{x}$ for $t \geq 0$ along the optimal program $\{x(t)\}$. Thus, $\{x(t)\}$ is monotonically decreasing for $t \geq 0$. By the turnpike property of optimal programs, (ii) follows. □

6. The optimal policy function under limited concavity

One of the ways in which we can view the results on the OPF, in the case of a linear welfare function, obtained in Khan and Mitra (2006), is to consider the nature of the OPF in the case of a strictly concave welfare function, but with "limited" degree of concavity. We do so in this section, where it is shown that, compared with Proposition 6, the range of the optimal policy function can be reduced substantially when the welfare function has limited concavity, in a sense that can be made precise. This result is obtained without restricting the value of the parameter ξ. It has the implication that when $\xi \geq 1$, and the welfare function has limited concavity, then the OPF is given precisely by the "pan map" obtained in the case of a linear utility function in Khan and Mitra (2006).

Let us define a function $g : X \to X$ as follows:

$$g(x) = \begin{cases} \dfrac{1}{a} - \xi x & \text{for } x \in A \\ \hat{x} & \text{for } x \in \left(\hat{x}, \dfrac{\hat{x}}{1-d}\right) \\ (1-d)x & \text{for } x \in \left[\dfrac{\hat{x}}{1-d}, \dfrac{1}{ad}\right] \end{cases}$$

Proposition 11. *The optimal policy function $h : X \to X$ satisfies:*

$$h(x) \leq g(x) \quad \text{for all } x \in X$$

provided:

$$w'(1)(1+ad) \geq w'(\hat{x}) \tag{50}$$

Proof. Suppose, on the contrary, there is some $x \in X$, such that $h(x) > g(x)$. Since $h(x) = g(x)$ for all $x \in A$, we must have $h(x) > g(x)$ for some $x \in D$, where $D = (\hat{x}, 1/ad]$. Denote the optimal program from x by $\{x(t), y(t)\}$. Pick $\varepsilon > 0$ such that $x' \equiv h(x) - \varepsilon > g(x)$. Then, we have $x' > (1-d)x$, and we can define $y' = 1 - a[x' - (1-d)x]$ and note that $y' > 1 - a[h(x) - (1-d)x] = y(0)$. Thus, $(x, x') \in \Omega$ and $y' \in \Lambda(x, x')$.

Now, we can write:

$$\begin{aligned} V(x) = \;& u(x, h(x)) + V(h(x)) \\ &= w(y(0)) - w(y') + w(y') + V(x') + V(h(x)) - V(x') \\ &\quad < w'(y')(-a\varepsilon) + V(x) + V(h(x)) - V(x') \\ &\quad < w'(1)(-a\varepsilon) + V(x) + m\hat{q}\varepsilon \\ &\leq a\varepsilon[m\hat{y} - w'(1)] + V(x) \\ &\leq V(x) \end{aligned}$$

the last inequality following from (50). This contradiction establishes the proposition. □

Remark:. We refer to condition (50) as "limited concavity" of the welfare function. Clearly, condition (50) holds when the degree of concavity of w is relatively small.

We can elaborate a bit on the notion of limited concavity expressed in (50). This can be written as:

$$w'(1)1 \geq w'(\hat{x})\hat{x} \tag{51}$$

Clearly, a sufficient condition for (51) to hold is that:

$$w'(y)y \text{ is non-decreasing in } y \text{ on } \mathbb{R}_{++} \tag{52}$$

Since, w is twice continuously differentiable, (52) holds if and only if:

$$\frac{(-w''(y))y}{w'(y)} \leq 1 \text{ for all } y \in \mathbb{R}_{++} \tag{53}$$

Condition (53) states that the elasticity of marginal welfare does not exceed unity. It is legitimate to measure the degree of concavity of w at a point by the elasticity of marginal welfare at that point, and so our informal use of the phrase "limited degree of concavity" of the welfare function should be understood to correspond formally to the inequality expressed in (53).

For the class of welfare functions, given by:

$$w(y) = y^{\alpha} \text{ where } \alpha \in (0, 1) \tag{54}$$

we have the elasticity of marginal welfare equal to $(1 - \alpha)$, and so (53) is always satisfied, and so the "limited concavity" condition (50) is also satisfied. This indicates the scope of application of Proposition 11.

6.1. The OPF when $\xi \geq 1$

We can now use Propositions 9–11 to examine the case where the parameter $\xi \geq 1$, and there is limited concavity of the welfare function. One would expect that the OPF in the strictly concave case would approach the OPF in the linear case (when the optimal policy correspondence in the linear case is actually a function) as the concavity of the welfare function is decreased. However, the next result tells us that the approach is not asymptotic. As soon as condition (50) holds, the OPF becomes the "pan map" and thereafter it stays the pan map, determined completely by the technological parameters of the model, and invariant to further decreases in the concavity of the welfare function. In particular, we obtain the rather surprising result that *for the entire class of welfare functions given by* (54), the OPF is described precisely by the "pan map" when $\xi \geq 1$.

Proposition 12. *When* $\xi \geq 1$, *the optimal policy function* $h : X \to X$ *satisfies*:

$$h(x) = \begin{cases} \dfrac{1}{a} - \xi x & \text{for } x \in A \\ \hat{x} & \text{for } x \in \left(\hat{x}, \dfrac{\hat{x}}{1-d}\right) \\ (1-d)x & \text{for } x \in \left[\dfrac{\hat{x}}{1-d}, \dfrac{1}{ad}\right] \end{cases}$$

provided:

$$w'(1)(1+ad) \geq w'(\hat{x})$$

Proof. When $x \in A$, we know, from Proposition 6, that $h(x) = (1/a) - \xi x$. For $x \in (\hat{x}, \hat{x}/(1-d))$, we know that $h(x) \leq \hat{x}$ from Proposition 11, and we also know that $h(x) \geq \hat{x}$ from Propositions 9 and 10(since $\xi \geq 1$). Thus, $h(x) = \hat{x}$ for all $x \in (\hat{x}, \hat{x}/(1-d))$. For $x \in [\hat{x}/(1-d), 1/ad]$, we know that $h(x) \leq (1-d)x$ by Proposition 11, and we also know that $h(x) \geq (1-d)x$ by Proposition 6. Thus, we have $h(x) = (1-d)x$ for all $x \in [\hat{x}/(1-d), 1/ad]$. □

6.2. The OPF when $\xi < 1$

We can now state the following result for the case of $\xi < 1$. Notice that unlike the linear welfare function case, where the OPF would be the check map, we are only able to say that the OPF will lie "between the pan and the check maps" when the welfare function is strictly concave with limited concavity. In other words, unlike in the linear welfare function case, the transition from the pan map to the check map appears to take place in the range of parameter values $\xi < 1$, rather than for the parameter value $\xi = 1$.

Proposition 13. *When$\xi < 1$, the optimal policy function $h : X \to X$ satisfies*:

$$h(x) \in \begin{cases} \left\{ \dfrac{1}{a} - \xi x \right\} & \text{for } x \in A \\ \left[\dfrac{1}{a} - \xi x, \hat{x} \right] & \text{for } x \in (\hat{x}, 1) \\ [(1-d)x, \hat{x}] & \text{for } x \in \left[1, \dfrac{\hat{x}}{1-d}\right] \\ \{(1-d)x\} & \text{for } x \in \left[\dfrac{\hat{x}}{1-d}, \dfrac{1}{ad}\right] \end{cases}$$

provided:

$$w'(1)(1+ad) \geq w'(\hat{x})$$

Proof. When $x \in A$, we know, from Proposition 6, that $h(x) = (1/a) - \xi x$. For $x \in (\hat{x}, 1]$, we know from Proposition 11 that $h(x) \leq \hat{x}$, and we also know from Proposition 6 that $h(x) \geq (1/a) - \xi x$, so that $h(x) \in [(1/a) - \xi x, \hat{x}]$. For $x \in [1, \hat{x}/(1-d))$, we know from Proposition 11 that $h(x) \leq \hat{x}$, and we also know from Proposition 6 that $h(x) \geq (1-d)x$, so that $h(x) \in [(1-d)x, \hat{x}]$. Finally, for $x \in [\hat{x}/(1-d), 1/ad]$, we know from Proposition 11 that $h(x) \leq (1-d)x$, and we also know from Proposition 6 that $h(x) \geq (1-d)x$, so that $h(x) = (1-d)x$. □

Condition (50), used in Proposition 13 is the same as that used in Proposition 12. And we have seen that this condition is always satisfied for the class of welfare functions given by (54). Thus, for this class of welfare functions, we have successfully narrowed down the range of the optimal policy function to *precisely the range of the optimal policy correspondence identified in* Khan and Mitra (2006) *in the case of the linear welfare function* (Fig. 2).

We can, however, say a bit more than Proposition 13 does in the case $\xi < 1$, for the domains $(\hat{x}, 1)$ and $[1, \hat{x}/(1-d))$, if we further restrict the concavity of the welfare function. In order to do this, we need a bound on the slope of the value function for the domain $(1-d, \hat{x})$, which Lemma 1 does not provide. To this end, we first establish the following Lemma.

Lemma 3. *Let $\xi < 1$, $x \in (1-d, \hat{x}]$ and $\varepsilon \in (0, x-(1-d))$. Define*:

$$x(t) = H^t(x),\ x'(t) = H^t(x-\varepsilon) \text{ forall } t \geq 0$$

and

$$y(t) = 1 - a[x(t+1) - (1-d)x(t)] \text{ for all } t \geq 0;\ y'(t) = 1 - a[x'(t+1) - (1-d)x'(t)] \text{ for all } t \geq 0$$

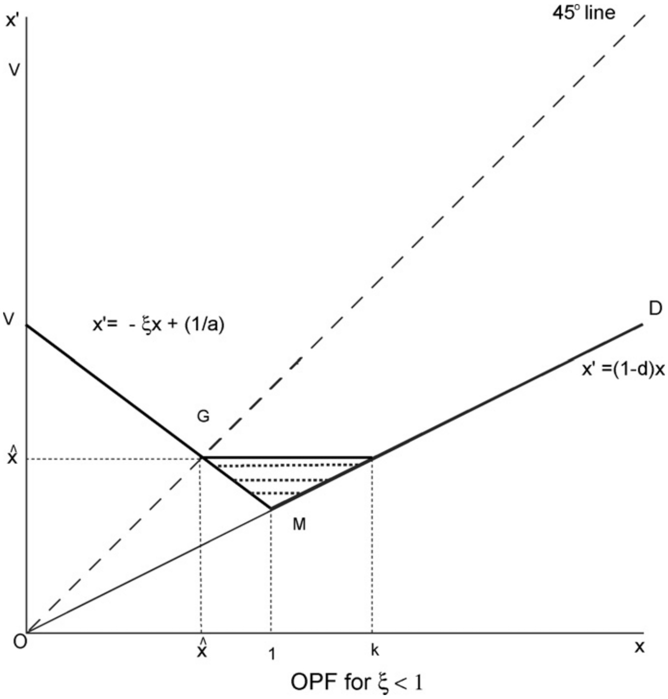

Fig. 2. Optimal policy function (limited concavity).

Then, we have:

$$\left.\begin{array}{ll} 0 \le w(y(t)) - w(y'(t)) \le w'(1-d)\xi^t\varepsilon & \textit{for } t = 0, 2, 4, \ldots \\ w(y'(t)) - w(y(t)) \ge w'(1)\xi^t\varepsilon & \textit{for } t = 1, 3, 5, \ldots \end{array}\right\} \tag{55}$$

and:

$$\sum_{t=0}^{\infty}[w(y(t)) - w(y'(t))] \le \frac{\varepsilon\left[w'(1-d) - \xi w'(1)\right]}{(1-\xi^2)} \tag{56}$$

Proof. First, note that $\{x(t)\}$ and $\{x'(t)\}$ are well-defined, and $x(t) \in (1-d, 1)$ and $x'(t) \in (1-d, 1)$ for all $t \ge 0$. Further, $x(t) \to \hat{x}$ and $x'(t) \to \hat{x}$ as $t \to \infty$. Second, the sequences $\{y(t)\}$ and $\{y'(t)\}$ are well-defined and $y(t) = x(t)$ and $y'(t) = x'(t)$ for all $t \ge 0$.

Since H has a negative slope, we have $y(t) = x(t) > x'(t) = y'(t)$ for $t = 0, 2, 4, \ldots$, while $y(t) = x(t) < x'(t) = y'(t)$ for $t = 1, 3, 5, \ldots$. Further, we have:

$$[x'(t+1) - x(t+1)] = \xi[x(t) - x'(t)] \text{ for all } t \ge 0$$

Thus, we obtain:

$$\begin{array}{ll} [y(t) - y'(t)] = \xi^t\varepsilon & \text{for } t = 0, 2, 4, \ldots \\ [y'(t) - y(t)] = \xi^t\varepsilon & \text{for } t = 1, 3, 5, \ldots \end{array}$$

Then, by noting that $y'(t) \in (1-d, 1)$, (55) follows from concavity of w. Using (55) and $\xi < 1$, it follows that the sum on the left-hand side of (56) is well-defined, and (56) follows by summing over the inequalities in (55). □

We are now in a position to provide a sufficient condition under which the OPF coincides with H on the domain $[\hat{x}, 1]$. This sufficient condition can be stated as follows:

$$w'(1)(\xi + a(1-\xi^2)) \ge w'(1-d)$$

Note that $(\xi + a(1-\xi^2)) = (\xi + a(1-\xi)(1+\xi)) = (\xi + (1-\xi)(1+ad)) < (1+ad)$ since $\xi \in (0,1)$. Thus, this is a more demanding condition than condition (50) used in Propositions 11–13. Also, note that $(\xi + a(1-\xi^2)) > 1$, so the condition will be satisfied if w is sufficiently close to being linear. In particular, for the class of welfare functions given by (54), the condition is satisfied for all α sufficiently close to 1, and for this case, we can in fact compute an explicit bound.

Proposition 14. *Let* $\xi < 1$, *and suppose the welfare function satisfies the condition*:

$$w'(1)(\xi + a(1-\xi^2)) \geq w'(1-d) \tag{57}$$

Then the optimal policy function $h : X \to X$ *satisfies*:

$$h(x) = \begin{cases} \dfrac{1}{a} - \xi x & \text{for } x \in A \\ \dfrac{1}{a} - \xi x & \text{for } x \in (\hat{x}, 1) \\ (1-d)x & \text{for } x \in \left[1, \dfrac{\hat{x}}{1-d}\right] \\ (1-d)x & \text{for } x \in \left[\dfrac{\hat{x}}{1-d}, \dfrac{1}{ad}\right] \end{cases}$$

Proof. We claim that under the condition (57), we must have $h(1) = (1-d)$. If this is not the case, then by Proposition 13, we have $h(1) \in ((1-d), \hat{x}]$. There are two cases to consider: (i) $h(1) = \hat{x}$; (ii) $h(1) \in ((1-d), \hat{x})$.

In case (i), note that $(1, \hat{x}, \hat{x}, \hat{x}, \ldots)$ is the optimal program from 1. Pick $\varepsilon \in (0, \hat{x} - (1-d))$, and define $x'(1) = \hat{x} - \varepsilon$, so that $x'(1) \in (1-d, \hat{x})$. Further, for $t \geq 2$, define $x'(t) = H^{t-1}(x'(1))$. Further, define:

$$\begin{aligned} y(t) &= 1 - a[x(t+1) - (1-d)x(t)] \quad \text{for all } t \geq 0 \\ y'(t) &= 1 - a[x'(t+1) - (1-d)x'(t)] \quad \text{for all } t \geq 0 \end{aligned}$$

Then, by applying Lemma 3, we have:

$$V(\hat{x}) - V(\hat{x} - \varepsilon) \leq \frac{\varepsilon\left[w'(1-d) - \xi w'(1)\right]}{(1-\xi^2)} \tag{58}$$

This allows us to make the following compuations:

$$\begin{aligned} V(1) &= u(1, \hat{x}) + V(\hat{x}) \\ &= w(y(0)) - w(y'(0)) + w(y'(0)) + V(x'(1)) + V(\hat{x}) - V(x'(1)) \\ &\quad < w'(y'(0))(-a\varepsilon) + V(1) + V(\hat{x}) - V(\hat{x} - \varepsilon) \\ &\quad < w'(1)(-a\varepsilon) + V(1) + \left\{\frac{\varepsilon[w'(1-d) - \xi w'(1)]}{1-\xi^2}\right\} \\ &= \varepsilon\left\{\frac{[w'(1-d) - w'(1)(\xi + a(1-\xi^2))]}{1-\xi^2}\right\} + V(1) \\ &\leq V(1) \end{aligned}$$

the second inequality following from (58) and the last inequality from (57). This is clearly a contradiction.

In case (ii), we have $h(1) \in ((1-d), \hat{x})$. Let $\tilde{x} = \sup\{x \in [\hat{x}, 1] : h(x) = H(x)\}$. Note that since $h(\hat{x}) = H(\hat{x})$, $\tilde{x}$ is well-defined, and since $h(1) > H(1)$, we have $\tilde{x} < 1$, by continuity of h and H. If $\tilde{x} = \hat{x}$, then $h(x) > H(x)$ for all $x \in (\hat{x}, 1)$ by Proposition 13. Then by Proposition 8, we have $D_+h(x) \geq 0$ for all $x \in (\hat{x}, 1)$. Thus, using the continuity of h, we have $h(1) \geq h(\hat{x})$ (see Royden, 1988, Proposition 2, p. 99). This is a contradiction since in case (ii), $h(1) < \hat{x} = h(\hat{x})$. Thus, we must have $\tilde{x} \in (\hat{x}, 1)$, and $h(\tilde{x}) = H(\tilde{x})$ by continuity of h and H while $h(x) > H(x)$ for all $x \in (\tilde{x}, 1)$. By Corollary 1, we have $h(x) = H(x)$ for all $x \in [\hat{x}, \tilde{x}]$.

Since $\xi \in (0,1)$, $\hat{x} < H^2(\tilde{x}) < \tilde{x}$, and so we can pick $x' \in (\tilde{x}, 1)$ sufficiently close to $\tilde{x}$ so that $H^2(x') < \tilde{x}$, and $h(x') \in (H(x'), \hat{x})$. Define $\varepsilon = h(x') - H(x')$. Then, we have $h^2(x') = H(h(x')) < H^2(x') < \tilde{x}$, and so

$\{x(t)\} = (x', h(x'), H(h(x')), H^2(h(x')), \ldots)$ is the optimal program from x'. Define $x'(t) = H^t(x')$ for $t \geq 0$, and:

$$y(t) = 1 - a[x(t+1) - (1-d)x(t)] \quad \text{for all } t \geq 0$$
$$y'(t) = 1 - a[x'(t+1) - (1-d)x'(t)] \quad \text{for all } t \geq 0$$

Then, by applying Lemma 3, we have:

$$V(h(x')) - V(h(x') - \varepsilon) \leq \frac{\varepsilon\left[w'(1-d) - \xi w'(1)\right]}{(1-\xi^2)} \tag{59}$$

This allows us to make the following compuations:

$$\begin{aligned} V(x') = \;& u(x', h(x')) + V(h(x')) \\ &= w(y(0)) - w(y'(0)) + w(y'(0)) + V(H(x')) + V(h(x')) - V(H(x')) < w'(y'(0))(-a\varepsilon) \\ &\quad + V(x') + V(h(x')) - V(h(x') - \varepsilon) < w'(1)(-a\varepsilon) + V(x') + \left\{\frac{\varepsilon[w'(1-d) - \xi w'(1)]}{1-\xi^2}\right\} \\ &= \varepsilon\left\{\frac{[w'(1-d) - w'(1)(\xi + a(1-\xi^2))]}{1-\xi^2}\right\} + V(x') \\ &\leq V(x') \end{aligned}$$

the second inequality following from (59) and the last inequality from (57). This is clearly a contradiction.

This establishes our claim that $h(1) = (1-d)$. By Corollary 1, we have $h(x) = (1/a) - \xi x$ for all $x \in [\hat{x}, 1]$.

We claim next that $h(x) = (1-d)x$ for all $x \in (1, k)$, where $k = \hat{x}/(1-d)$. If this were not true, then by Proposition 13, there is some $x' \in (1, k)$, such that $h(x') \in ((1-d)x', \hat{x}]$. If $h(x') = \hat{x}$, then we can follow the proof of case (i) above, and if $h(x') \in ((1-d), \hat{x})$ then we can follow the proof of case (ii) above, to arrive at a contradiction. This establishes the claim that $h(x) = (1-d)x$ for all $x \in (1, k)$. Given Proposition 13, the proof is now complete. □

As a final result, dealing with the case of $\xi \in (0, 1)$, we provide an explicit bound on the elasticity of marginal welfare for the class of welfare functions given by (54), which ensures that the OPF is the check map.

To this end, denote $[\xi + a(1-\xi^2)]$ by Q. As already noted above, we have:

$$Q = (\xi + a(1-\xi^2)) = (\xi + a(1-\xi)(1+\xi)) = (\xi + (1-\xi)(1+ad))$$

Since $\xi \in (0, 1)$, we have $Q \in (1, (1+ad))$, and so:

$$Q(1-d) < (1+ad)(1-d) = 1 - ad\xi < 1 \tag{60}$$

Define:

$$\beta = \frac{\ln Q}{\ln[1/(1-d)]} \tag{61}$$

Using (60), we clearly have $\beta \in (0, 1)$.

Corollary 2. *Let* $\xi < 1$, *and let the welfare function*, w, *be given by*:

$$w(y) = y^{\alpha} \text{ for } y \geq 0$$

where $\alpha \in (0, 1)$. *Suppose the elasticity of marginal welfare*, $(1-\alpha)$ *satisfies*:

$$(1-\alpha) \leq \beta \tag{62}$$

where β is defined by (61). Then the optimal policy functionh : $X \to X$ *satisfies*:

$$h(x) = \begin{cases} \dfrac{1}{a} - \xi x & \textit{for } x \in A \\ \dfrac{1}{a} - \xi x & \textit{for } x \in (\hat{x}, 1) \\ (1-d)x & \textit{for } x \in \left[1, \dfrac{\hat{x}}{1-d}\right] \\ (1-d)x & \textit{for } x \in \left[\dfrac{\hat{x}}{1-d}, \dfrac{1}{ad}\right] \end{cases}$$

Proof. Given Proposition 14, it is enough to verify that given (62), condition (57) is satisfied. Given the class of welfare functions, condition (57) can be written as:

$$\alpha(\xi + a(1-\xi^2)) \geq \frac{\alpha}{(1-d)^{1-\alpha}}$$

which is satisfied iff:

$$\ln Q \geq (1-\alpha)\ln\left[\frac{1}{1-d}\right]$$

This condition is precisely (62). □

7. Concluding remark

For the class of welfare functions given by (54), we make the following observation. For $a = 1$ and $d = (1/2)$, we have $\xi = (1/2)$, and condition (62) is satisfied for $\alpha \geq 0.68$. Thus, for $\alpha = 0.7$, we get the check-map as the OPF. If we change a to (2/3), we get $\xi = 1$, and the OPF for $\alpha = 0.7$ (in fact, for all $\alpha \in (0, 1)$) is the pan map. Thus, one might say that as a changes from 1 to (2/3), the parameters d and α fixed at (1/2) and 0.7 respectively, the OPF is transformed from a check map to a pan map. The exact process of transformation is a topic worth studying, and involves a deeper analysis of the OPF, as a function of the parameters of the model. We leave this topic for future research.

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